McGill University, Montreal
GEOG 321 - Climatic Environments
Knox

Answers to Study Questions - Topic 10

1. $C = \rho c_p$, where ρ is the density of the material. For water we know that $\rho = 1 \times 10^3\,\mathrm{kg\,m^{-3}}$ (i.e. $1\,\mathrm{kg} = 1\ell = 10 \times 10 \times 10\,\mathrm{cm}$), therefore $4.180 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}} \times 10^3\,\mathrm{kg\,m^{-3}} = \underline{4.180 \times 10^6\,\mathrm{J\,m^{-3}\,K^{-1}}}$.

2. $P = 55\%$ for a dry soil means $\theta_a = 0.55$ and $\theta_m = 1 - \theta_a = 0.45$:

$$\begin{aligned} C &= \theta_m C_m + \theta_a C_a \\ &\approx \theta_m C_m \\ &= 0.45 \times 2.1\,\mathrm{MJ\,m^{-3}\,K^{-1}} \\ &= \underline{0.945\,\mathrm{MJ\,m^{-3}\,K^{-1}}} \end{aligned}$$

   Subscripts a and m refer to air and mineral matter, respectively. Note, the second term is small compared to the first one ($\theta_a C_a = 0.45 \times 0.0012\,\mathrm{MJ\,m^{-3}\,K^{-1}} = 0.00066\,\mathrm{MJ\,m^{-3}\,K^{-1}}$) and can be neglected.

3. $P = 55\%$ for the saturated case means $\theta_w = 0.55$ and $\theta_m = 1 - \theta_w = 0.45$:

$$\begin{aligned} C &= \theta_m C_m + \theta_w C_w \\ &= 0.45 \times 2.1\,\mathrm{MJ\,m^{-3}\,K^{-1}} + 0.55 \times 4.18\,\mathrm{MJ\,m^{-3}\,K^{-1}} \\ &= 0.945\,\mathrm{MJ\,m^{-3}\,K^{-1}} + 2.299\,\mathrm{MJ\,m^{-3}\,K^{-1}} \\ &= \underline{3.24\,\mathrm{MJ\,m^{-3}\,K^{-1}}} \end{aligned}$$

   Subscripts a and m refer to air and mineral matter, respectively. Note, the second term is small compared to the first one ($\theta_a C_a = 0.45 \times 0.0012\,\mathrm{MJ\,m^{-3}\,K^{-1}} = 0.00066\,\mathrm{MJ\,m^{-3}\,K^{-1}}$) and can be neglected.

4. $P = 50\%$ and $\theta_a = 0.30$ means $\theta_w = P - \theta_a = 0.20$. An organic to mineral ratio of 1.5 (3/2) means $(1 - P) = 0.5$ is made up of 0.3 θ_o and 0.2 θ_m:

$$\begin{aligned} C &= \theta_m C_m + \theta_o C_o + \theta_w C_w \\ &= 0.2 \times 2.5\,\mathrm{MJ\,m^{-3}\,K^{-1}} + 0.3 \times 2.1\,\mathrm{MJ\,m^{-3}\,K^{-1}} \\ &\quad + 0.2 \times 4.18\,\mathrm{MJ\,m^{-3}\,K^{-1}} \\ &= \underline{1.97\,\mathrm{MJ\,m^{-3}\,K^{-1}}} \end{aligned}$$

5. $\Delta\theta_w = 0.1$:

$$\begin{aligned} \Delta C &= \Delta\theta_w \, C_w \\ &= 0.1 \times 4.18\,\mathrm{MJ\,m^{-3}\,K^{-1}} \\ &= \underline{0.418\,\mathrm{MJ\,m^{-3}\,K^{-1}}} \end{aligned}$$

   C of the soil will increase by $0.418\,\mathrm{MJ\,m^{-3}\,K^{-1}}$.

6. The warming rate of a material is defined by:

$$\frac{\Delta T}{\Delta t} = \frac{1}{C}\,\frac{\Delta Q_G}{\Delta z}$$

   Q_G at 0 cm depth (surface) is $+100\,\mathrm{W\,m^{-3}}$, Q_G at 10 cm must be zero because there is no energy distributed to lower layers, i.e. only topmost 10 cm experience heating, hence $\Delta Q_G = 100\,\mathrm{W\,m^{-2}} - 0\,\mathrm{W\,m^{-2}} = 100\,\mathrm{W\,m^{-2}}$ (same as $\mathrm{J\,s^{-1}\,m^{-2}}$):

$$\frac{\Delta T}{\Delta t} = \frac{100\,\mathrm{J\,s^{-1}\,m^{-2}}}{2\,\mathrm{MJ\,m^{-3}\,K^{-1}} \times 0.1\mathrm{m}} = 0.0005\,\mathrm{K\,s^{-1}} = \underline{1.8\,\mathrm{K\,h^{-1}}}.$$

7. Fourier's Law: $Q_G = -k\,\Delta T/\Delta z = -k(T_2 - T_1)/(z_2 - z_1)$. It states that the flow rate of heat conducted through a solid material (or still fluid) is proportional to the temperature gradient.

8. Use Fourier's Law:

$$Q_G = -k\,\frac{\Delta T}{\Delta z}$$

   You insert the thermal conductivity of $k = 0.27\,\mathrm{W\,m^{-1}\,K^{-1}}$:

$$\begin{aligned} Q_G &= -0.27\,\mathrm{W\,m^{-1}\,K^{-1}}\,\frac{20^\circ\mathrm{C} - 18.5^\circ\mathrm{C}}{0.02\,\mathrm{m} - 0.06\,\mathrm{m}} \\ &= -0.27\,\mathrm{W\,m^{-1}\,K^{-1}}\,\frac{1.5\mathrm{K}}{-0.04\,\mathrm{m}} \\ &= \underline{10.1\,\mathrm{W\,m^{-2}}} \end{aligned}$$

9. Again we use Fourier's Law, but rearranged:

$$-Q_G\frac{\Delta z}{\Delta T} = k$$

   We can directly plug-in the inverse of the gradient ($\Delta T/\Delta z = -0.5\,\mathrm{K\,cm^{-1}}$) into $\Delta z/\Delta T$:

$$\begin{aligned} k &= -20\,\mathrm{W\,m^{-2}} \times -0.02\,\mathrm{K\,m^{-1}} \\ &= \underline{+0.4\,\mathrm{W\,m^{-1}\,K^{-1}}} \end{aligned}$$

10. The thermal diffusivity κ tells us how quickly temperature waves propagate down into the soil, and κ is defined by:

$$\begin{aligned}\kappa &= \frac{k}{C} = \frac{k}{\rho\, c_p} \\ &= \frac{0.4\,\mathrm{J\,s^{-1}\,m^{-1}\,K^{-1}}}{1.4\times10^{3}\,\mathrm{kg\,m^{-3}}\,1.8\times10^{3}\,\mathrm{J\,kg^{-1}\,K^{-1}}} \\ &= \underline{0.16\times10^{-6}\,\mathrm{m^2\,s^{-1}}}\end{aligned}$$

Note the fine - but important - difference between the symbol k for thermal conductivity (Latin k') and the symbol κ for thermal diffusivity (Greek 'kappa').

11. The thermal admittance μ is strictly speaking a surface property. It defines how well a surface can accept or release heat.

$$\begin{aligned}\mu &= \sqrt{k\,C} = \sqrt{k\,\rho\,c_p} \\ &= \sqrt{0.4\,\mathrm{J\,s^{-1}\,m^{-1}\,K^{-1}}\times1.4\times10^{3}\,\mathrm{kg\,m^{-3}}\times1.8\times10^{3}\,\mathrm{J\,kg^{-1}\,K^{-1}}} \\ &= \sqrt{1.008\times10^{6}\,\mathrm{J^2\,m^{-4}\,K^{-2}\,s^{-1}}} \\ &= \underline{1004\,\mathrm{J\,m^{-2}\,K^{-1}\,s^{-1/2}}}\end{aligned}$$

12. M_m and M_o is the mass of mineral and organic material, respectively, in one m^3 of soil.

    The total mass of the dry soil M in one cubic-metre is given by the bulk density ($\rho_s = 1.4\mathrm{Mg\,m^{-3}}$):

$$M = M_m + M_o = \rho_s \times 1\,\mathrm{m}^3 = 1.4\,\mathrm{Mg\,m^{-3}} \times 1\,\mathrm{m}^3 = 1.4\,\mathrm{Mg}$$

    f_o is the organic mass fraction (given: $f_o = 0.25$) which is the mass of organic material to the total mass

$$f_o = \frac{M_o}{M} = \frac{M_o}{M_o + M_m} = 0.25$$

    solving for M_m and M_o:

$$M_m = M \times (1 - f_o) = 1.4\,\mathrm{Mg} \times (1 - 0.25) = 1.4\,\mathrm{Mg}\times)0.75 = 1.05\,\mathrm{Mg}$$

$$M_o = M \times f_o = 1.4\,\mathrm{Mg} \times 0.25 = 0.35\,\mathrm{Mg}$$

13. Using the mass of organic and mineral material contained in one cubic metre (determined in Question 12), we can formulate the densities of organic (ρ_o) and mineral material (ρ_m) in the same soil:

$$\rho_o = \frac{M_o}{1\mathrm{m}^3 \times \theta_\mathrm{o}}$$

$$\rho_m = \frac{M_m}{1\mathrm{m}^3 \times \theta_\mathrm{m}}$$

Where ($1\mathrm{m}^3 \times \theta_\mathrm{o}$) is the volume of organic material in one cubic metre, and ($1\mathrm{m}^3\theta_\mathrm{m}$) is the volume of mineral material in one cubic metre.

Lecture 10, slide 5 provides $c_m = 0.8\mathrm{J\,kg^{-1}\,K^{-1}}$ and $c_o = 1.9\mathrm{J\,kg^{-1}\,K^{-1}}$. Using $C = \rho c$ allows us to then determine the heat capacity of organic (C_o) and mineral material (C_m) in this soil:

$$C_o = \rho_o\, c_o$$

$$C_m = \rho_m\, c_m$$

The composite heat capacity of the dry soil (C_s) is the sum of the compound heat capacities weighted by the respective volume fractions:

$$C_s = \theta_o C_o + \theta_m C_m$$

replacing C_o by $\rho_o\, c_o$ (and same for C_m) then gives:

$$C_s = \theta_o \rho_o\, c_o + \theta_m \rho_m\, c_m$$

replacing ρ_o by $\frac{M_o}{1\mathrm{m}^3\theta_\mathrm{o}}$ (and same for ρ_m) then gives:

$$C_s = \theta_o \frac{M_o}{1\mathrm{m}^3\theta_\mathrm{o}}\, c_o + \theta_m \frac{M_m}{1\mathrm{m}^3\theta_\mathrm{m}}\, c_m$$

Note that then θ_o and θ_m cancel out:

$$C_s = \frac{M_o}{1\mathrm{m}^3}\, c_o + \frac{M_m}{1\mathrm{m}^3}\, c_m$$

Inserting the values:

$$C_s = \frac{0.35\,\mathrm{Mg}}{1\mathrm{m}^3}\,1.9\,\mathrm{kJ\,kg^{-1}\,K^{-1}} + \frac{1.05\,\mathrm{Mg}}{1\mathrm{m}^3}\,0.8\,\mathrm{kJ\,kg^{-1}\,K^{-1}} =$$

$$0.665\,\mathrm{MJ\,m^{-3}\,K^{-1}} + 0.84\,\mathrm{MJ\,m^{-3}\,K^{-1}} = 1.505\,\mathrm{MJ\,m^{-3}\,K^{-1}}$$

We learn from this exercise that generally we can rewrite the composite heat capacity of a soil (C_s) from using heat capacities of compound substances and volume fractions:

$$C_s = \theta_o C_o + \theta_m C_m + ...$$

to using specific heat and known mass (M_o, M_m, etc.) of the compound substances in a soil volume V_s:

$$C_s = \frac{M_o}{V_s}\,c_o + \frac{M_o}{V_s}\,c_m + ...$$

where V_s is the volume of the soil, and M_o and M_m is the mass of organic and mineral material in the same volume V_s. This has the advantage of avoiding assuming any specific denisty of organic and mineral material, which is difficult to determine practically.